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Abstract

Siegel (1995) has developed a technique with which the systematic risk of a security (beta) can be estimated without recourse to historical capital market data. Instead, beta is estimated implicitly from the current market prices of exchange options that enable the exchange of a security against shares on the market index. Because this type of exchange options is not currently traded on the capital markets, Siegel's technique cannot yet be used in practice. This article will show that beta can also be estimated implicitly from the current market prices of plain vanilla options, based on the Capital Asset Pricing Model.

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1 Introduction

The Capital Asset Pricing Model (CAPM) of *Sharpe* (1964) and *Lintner* (1965) continues to be of central importance to the valuation of risk-bearing securities, in theory as well as in practice. According to the CAPM, the expected rate of return on a security depends primarily on its systematic risk (beta), which is normally estimated by means of a regressive analysis of historical capital market data. Of all of the numerous empirical tests of the CAPM, the study by *Fama and French* (1992) in particular generated much attention. According to this study, beta has hardly any explanatory power for the expected rate of return on a security. In fact, the expected rate of return depends much more on the size of a company and the book-to-market ratio. *Berk* (1995) showed nonetheless that these effects can also be traced back to a flawed measurement of beta.[1] Statistical errors can be caused in particular by the fact that beta changes through time.[2] In order to avoid this problem, *Siegel* (1995) proposes a method with which beta can be estimated from current options prices, without recourse to historical capital market data. However, practical application of this method requires that exchange options be traded that entitle the exchange of securities for shares on the market index. Presently, such options are not traded on the capital market. The purpose of this paper is to universalize Siegel's method so that beta can also be estimated from plain vanilla options.

The *Siegel* (1995) method is based on estimation of implicit volatility according to *Latané and Rendleman* (1976), whose technique is considered the standard in option pricing today.[3] *Siegel* (1995) ties this technique together with the valuation of exchange options according to *Margrabe* (1978) in order to estimate implicit beta. *Siegel* (1997), *Campa and Chang* (1998), and *Walter and Lopez* (2000) use similar approaches to obtain implied correlation of currencies from currency options. Recently, *Skintzi and Refenes* (2005) propose a method in forecasting future index correlation called *implied correlation index* that is also based on current option prices.

In this article the *Siegel* (1995) method will be universalized in that the implicit density function of an underlying asset is estimated implicitly from the theoretical CAPM prices of plain vanilla options.[4] The beta of an underlying asset results from the moments of this density function. The theoretical basis for calculation of implicit, risk-neutral density functions originates from *Ross* (1976) and *Breeden and Litzenberger* (1978) and has been used in numerous works to this day: *Rubinstein* (1994), *Darman and Kani* (1994), *Jackwerth and Rubinstein* (1996), and *Brown and Toft* (1999) estimate implicit risk-neutral density functions with the help of a modified binomial model (Implied Binomial Trees). *Shimko* (1993), *Jarrow and Rudd* (1982), and *Longstaff* (1995) estimate the price functions of options directly from their observed market prices, in dependence on

[1] *Fama and French* (2004) discuss the empirical problems that may be caused by difficulties in implementing valid tests of the CAPM.

[2] *Skintzi and Refenes* (2005) and *Longin and Solnik* (2001), for example, observed that correlations of stocks returns increase in highly volatile or bear markets.

[3] See *Blair et al.* (2001) for recent studies on the predictive ability of implied volatility.

[4] *Dennis and Mayhew* (2002) investigated the relative importance of beta in explaining the prices of stock options traded on the Chicago Board Options Exchange.

the exercise price, and from there derive risk-neutral density functions. *Aït-Sahalia and Lo* (2000) and *Jackwerth* (2000) determine a clear difference between risk-neutral and subjective expectations and attempt to draw conclusions from this regarding the risk aversion of market participants. *Jackwerth* (2000) arrives furthermore at the result that the historical capital market rates of return are approximately lognormally distributed.

The notation and model assumptions are explained in section 2. In section 3, a model is presented with which calls can be evaluated based on the CAPM when rates of return are distributed lognormally. On this basis, it is possible to estimate beta implicitly from the prices of ordinary calls in section 4. Section 5 summarizes the results of the article.

2 Assumptions and Notation

The valuation of options in section 3 is based on the assumptions of the one-period CAPM:

1. Risk-averse investors maximize the μ-σ-utility of their end-of-period wealth.
2. Investors have homogeneous expectations about assets returns; the instantaneous rate of return on any asset and the market portfolio have a joint normal distribution.[5] Investors may borrow or lend unlimited amounts at the risk-free rate.
3. Markets are frictionless. Information is costless and simultaneously available to all investors. There are no market imperfections such as transaction costs, taxes, or restrictions on short selling.

The following notation is used throughout the paper:

K	Exercise price on an option
$p(\tilde{X}_c)$	Price of a call on an asset S with cashflow $\tilde{X}_c$
$p(\tilde{X}_{cm})$	Price of a call on the market index $\tilde{X}_m$ with cashflow $\tilde{X}_{cm}$
$p(\tilde{X}_{ce})$	Price of an exchange option call with cashflow $\tilde{X}_{ce}$
$p(\tilde{X}_s)$	Price of an underlying asset S with cashflow $\tilde{X}_s$
$p(\tilde{X}_m)$	Current Market index
n_s	Number of shares of asset S to be exchanged under the exchange option
n_m	Number of shares of the market index under the exchange option
$\tilde{R}_s$	Standardized cashflow of an underlying asset, $\tilde{R}_s = \tilde{X}_s \,/\, p(\tilde{X}_s)$
$\tilde{R}_m$	Standardized cashflow of the market portfolio, $\tilde{R}_m = \tilde{X}_m \,/\, p(\tilde{X}_m)$
β_s	Beta of an underlying asset S with respect to the market index
r_f	Instantaneous risk-free rate of interest
$\tilde{r}_s$	Instantaneous rate of return on asset S
$\tilde{r}_m$	Instantaneous rate of return on the market index
μ_s	Expected instantaneous rate of return on asset S

[5] For a definition of bivariate normal distribution, see Appendix A.

μ_m	Expected instantaneous rate of return on the market index
σ_S	Instantaneous variance of the rate of return on asset S
σ_m	Instantaneous variance of the rate of return on the market index
ρ	Instantaneous correlation between the rates of return on asset S and on the market index

In the case of the given parameters for bivariate normal distribution of rates of return, the following applies for the expected values, variances and covariances of the securities' cash flow and market portfolio's standardized cash flow[6]

$$E\,[\tilde{X}_s] = p(\tilde{X}_s)\, e^{\mu_s + \frac{1}{2}\sigma_s^2}, \tag{1}$$

$$E\,[\tilde{R}_m] = e^{\mu_m + \frac{1}{2}\sigma_m^2}, \tag{2}$$

$$Var\,[\tilde{R}_m] = e^{2\mu_m + \sigma_m^2}\left(e^{\sigma_m^2} - 1\right), \tag{3}$$

$$Cov\,[\tilde{X}_s, \tilde{R}_m] = p(\tilde{X}_s)\, e^{\mu_m + \frac{1}{2}\sigma_m^2 + \mu_s + \frac{1}{2}\sigma_s^2}\left(e^{\rho\,\sigma_m\sigma_s} - 1\right). \tag{4}$$

For the standard definition of beta, the following results in the case of bivariate normal distribution[7]

$$\beta_s = \frac{Cov\,[\tilde{R}_s, \tilde{R}_m]}{Var\,[\tilde{R}_m]} = \frac{e^{\mu_s + \frac{1}{2}\sigma_s^2} \cdot \left(e^{\rho\,\sigma_s\sigma_m} - 1\right)}{e^{\mu_m + \frac{1}{2}\sigma_m^2} \cdot \left(e^{\sigma_m^2} - 1\right)}. \tag{5}$$

3 The Model

3.1 Option Pricing in an Incomplete Lognormal Market

In an incomplete lognormal market the CAPM may be used for option pricing.[8] The well-known *certainty equivalent valuation formula* of the single-period CAPM is[9]

$$p(\tilde{X}_c) = \frac{E\,[\tilde{X}_c] - \lambda \cdot Cov\,[\tilde{X}_c, \tilde{R}_m]}{1 + r_f^*} \qquad \text{where} \quad \lambda = \frac{E\,[\tilde{R}_m] - (1 + r_f^*)}{Var\,[\tilde{R}_m]}. \tag{6}$$

In order to be able to apply this equation to the valuation of a call, the expected cash flow of the call and the covariance between the cash flow of the call and the rates of return on the market portfolio must first be determined. Under the assumption of lognormally

[6] The moments of lognormal distribution can be calculated with the help of the integrals (38), (39) and (40) indicated in Appendix A.

[7] For a general definition of beta, see *Copeland and Weston* (1988), pg. 198.

[8] Options are redundant securities in a complete market. However, the empirical results of *Vanden* (2004) indicate that options are nonredundant for explaining the returns on risky assets.

[9] See *Copeland and Weston* (1988), pg. 203.

distributed rates of return, we derive[10]

$$\begin{aligned}
E\,[\tilde{X}_c] &= p(\tilde{X}_s)\cdot e^{\mu_s+\frac{1}{2}\sigma_s^2}\cdot\Phi\,(d_1) - K\cdot\Phi\,(d_2)\,, && (7)\\
Cov\,[\tilde{X}_c,\tilde{R}_m] &= p(\tilde{X}_s)\cdot e^{\mu_s+\frac{1}{2}\sigma_s^2+\mu_m+\frac{1}{2}\sigma_m^2}\cdot\left(e^{\rho\,\sigma_s\sigma_m}\cdot\Phi\,(d_3)-\Phi\,(d_1)\right)\\
&\quad - K\cdot e^{\mu_m+\frac{1}{2}\sigma_m^2}\cdot\left(\Phi\,(d_4)-\Phi\,(d_2)\right), && (8)\\
d_1 &= (\ln(p(\tilde{X}_s)/K)+\mu_s)/(\sigma_s)+\sigma_s, && (9)\\
d_2 &= (\ln(p(\tilde{X}_s)/K)+\mu_s)/(\sigma_s), && (10)\\
d_3 &= (\ln(p(\tilde{X}_s)/K)+\mu_s)/(\sigma_s)+\sigma_s+\rho\,\sigma_m, && (11)\\
d_4 &= (\ln(p(\tilde{X}_s)/K)+\mu_s)/(\sigma_s)+\rho\,\sigma_m. && (12)
\end{aligned}$$

If we insert (7) and (8) in (6), after further conversion we get a representation that allows a comparison with the valuation equation according to *Black and Scholes* (1973)[11]

$$p(\tilde{X}_c) = p(\tilde{X}_s)\,\theta_1 - K\,e^{-r_f}\,\theta_2 \tag{13}$$

$$\text{where}\quad \theta_1 = e^{\mu_s+\frac{1}{2}\sigma_s^2-r_f}\Big(\Phi\,(d_1)-\lambda\,e^{\mu_m+\frac{1}{2}\sigma_m^2}\left(e^{\rho\sigma_s\sigma_m}\Phi\,(d_3)-\Phi\,(d_1)\right)\Big) \tag{14}$$

$$\theta_2 = \Phi\,(d_2)-\lambda\,e^{\mu_m+\frac{1}{2}\sigma_m^2}\left(\Phi\,(d_4)-\Phi\,(d_2)\right). \tag{15}$$

This model can be applied to the special case of complete markets. On complete markets, a risk-neutral valuation always leads to the correct valuation result.[12] In a risk-neutral world, the rate of return of the expected cash flow of a given risk-bearing financial title and that of the market portfolio equal the risk-free interest rate[13]

$$\mu_s + \sigma_s^2/2 = r_f\ , \tag{16}$$

$$\mu_m + \sigma_m^2/2 = r_f\ . \tag{17}$$

This correlation can also be intuitively justified. Market participants may only expect a risk premium for their risk-bearing financial title if they cannot nullify the risk through diversification of their portfolio. Because systematic risk can be nullified through diversification in complete markets, the market price of the risk is zero. From (16) and (17) follows $\lambda = 0$, $\theta_1 = \Phi\,(d_1)$ and $\theta_2 = \Phi\,(d_2)$. The valuation equation (13) is reduced accordingly with risk-neutral valuation to

$$p(\tilde{X}_c\,|\,\lambda=0) = p(\tilde{X}_s)\,\Phi\left(\frac{\ln(p(\tilde{X}_s)/K)+r_f+\sigma_s^2/2}{\sigma_s}\right) - Ke^{-r_f}\,\Phi\left(\frac{\ln(p(\tilde{X}_s)/K)+r_f-\sigma_s^2/2}{\sigma_s}\right)\ , \tag{18}$$

which equals the valuation equation of *Black and Scholes* (1973).

[10] See Appendix B. Put prices follow from put-call parity.

[11] *Ritchken* (1985a) developed a similar valuation equation for options based on the CAPM. This model is not consistent with the *Black and Scholes* (1973) model in the case of risk-neutral valuation, however.

[12] See *Cox and Ross* (1976).

[13] If the expected instantaneous rate of return on a security equals μ_s and the rate of return is lognormally distributed, then the rate of return on the expected cash flow equals $\mu_s + \sigma_s^2$.

3.2 Pricing Options on the Market Index

Jarrow and Madan (1997) developed a valuation model for calls on the market index that is also based on the assumptions of the CAPM and lognormally distributed rates of return. If we use the notation established above, then the value of a call on the market index equals[14]

$$p(\tilde{X}_{cm}) = p(\tilde{X}_m)\,\theta_{m1} - K\,e^{-r_f}\,\theta_{m2} \tag{20}$$

$$\text{where}\quad \theta_{m1} = e^{\mu_m+\frac{1}{2}\sigma_m^2-r_f}\left(\Phi\left(dm_1\right) - \lambda\,e^{\mu_m+\frac{1}{2}\sigma_m^2}\left(e^{\sigma_m^2}\Phi\left(dm_3\right) - \Phi\left(dm_1\right)\right)\right), \tag{21}$$

$$\theta_{m2} = \Phi\left(dm_2\right) - \lambda\,e^{\mu_m+\frac{1}{2}\sigma_m^2}\left(\Phi\left(dm_1\right) - \Phi\left(dm_2\right)\right), \tag{22}$$

$$dm_1 = (\ln(p(\tilde{X}_m)/K) + \mu_m + \sigma_m^2)\,/\,(\sigma_m)\,, \tag{23}$$

$$dm_2 = (\ln(p(\tilde{X}_m)/K) + \mu_m)\,/\,(\sigma_m)\,, \tag{24}$$

$$dm_3 = (\ln(p(\tilde{X}_m)/K) + \mu_m + 2\sigma_m^2)\,/\,(\sigma_m)\,. \tag{25}$$

This valuation equation is solely a special case of (13); for calls on the market index, the following apply: $\rho = 1$, $\mu_s = \mu_m$ and $\sigma_s = \sigma_m$.

4 Implicit Beta

4.1 Estimating Beta Using Exchange Options

Siegel (1995) assumes that continuous security trading on perfect capital markets is possible.[15] This standard assumption of options price theory enables a risk-neutral valuation of options and is equivalent to the assumption of complete capital markets.[16] Because the theoretical option prices in the case of risk-neutral valuation are independent of the correlation of cash flow of the underlying asset with that of the market portfolio, beta cannot be implicitly estimated from simple options. *Siegel* (1995) therefore recourses to

[14] *Jarrow and Madan* (1997) define the parameter μ_m as the rate of return of the expected value, while we use it to identify the expected rate of return. In order to establish comparability with our results, the parameter μ must be replaced with $\mu + \sigma^2/2$ in the work by *Jarrow and Madan* (1997),

$$p(\tilde{X}_{cm}) = (a + b\,K)\,p(\tilde{X}_m)\,e^{\mu_m+\frac{1}{2}\sigma_m^2}\,\Phi\left(dm_1\right) - a\,K\Phi\left(dm_2\right) - b\,p(\tilde{X}_m)^2\,e^{2(\mu_m+\sigma_m^2)}\,(dm_3) \tag{19}$$

$$\text{where}\quad a = (e^{\sigma_m^2-r_f} - e^{-(\mu_m+\frac{1}{2}\sigma_m^2)})\,/\,(e^{\sigma_m^2}-1)$$

$$b = (e^{\mu_m+\frac{1}{2}\sigma_m^2-r_f} - 1)\,/\,(p(\tilde{X}_m)\,e^{2(\mu_m+\sigma_m^2)}\,(e^{\sigma_m^2}-1))\,.$$

If we furthermore assume that the investor's planning horizon and the time to maturity of the option are identical, following elementary conversions, the valuation equation (20) results from the valuation equation (19). However, for the special case of calls on the market index, the *Ritchken* (1985a) model is not identical with the *Jarrow and Madan* (1997) model.

[15] See Assumption 1 in *Siegel* (1995).

[16] See *Cox et al.* (1979).

exchange options, which securitize the right for exchange of a financial title for shares on the market portfolio. The theoretical price of an exchange option in terms of risk-neutral valuation depends on the correlation of the cash flow of a financial title with the rates of return of the market portfolio and is therefore generally suitable for determining implicit beta factors. The risk-neutral valuation of exchange options is based on *Margrabe* (1978),

$$p(\tilde{X}_{ce}) = n_s\, p(\tilde{X}_s)\, \Phi\left(\frac{\ln\left(\frac{n_s\, p(\tilde{X}_s)}{n_m\, p(\tilde{X}_m)}\right) + \sigma_e^2/2}{\sigma_e}\right) - n_m\, p(\tilde{X}_m)\, \Phi\left(\frac{\ln\left(\frac{n_s\, p(\tilde{X}_s)}{n_m\, p(\tilde{X}_m)}\right) - \sigma_e^2/2}{\sigma_e}\right), \tag{26}$$

whereby the volatility σ_e depends on the volatilities of the underlying assets and the correlation of their rates of return,

$$\sigma_e^2 = Var\,[\tilde{r}_s - \tilde{r}_m] \;=\; \sigma_s^2 + \sigma_m^2 - 2\,\rho_{sm}\,\sigma_s\,\sigma_m\,. \tag{27}$$

Siegel (1995) assumes that three types of options are traded on the capital market: options on a common asset, options on the market index, and options that entitle the exchange of securities for shares on the market index. His idea for determination of implicit beta factors consists of first estimating the volatilities of the two underlying assets and the volatility σ_e of the exchange option implicitly from traded options. The correlation coefficient is then derived from correlation (27),

$$\rho_{sm} = (\sigma_s^2 + \sigma_m^2 - \sigma_e^2)\,/\,(2\,\sigma_s\,\sigma_m). \tag{28}$$

According to *Siegel* (1995), this results in the beta factor of the asset,

$$\beta_s^{Siegel} := \rho_{sm}\,\sigma_s\,/\,\sigma_m \;=\; (\sigma_s^2 + \sigma_m^2 - \sigma_e^2)\,/\,(2\,\sigma_m^2). \tag{29}$$

Leland (1999) describes definition (29) as modified beta. Even in risk-neutral valuation, this definition does not equal the standard definition of beta[17]

$$\beta_s = \frac{Cov\,[\tilde{R}_s, \tilde{R}_m]}{Var\,[\tilde{R}_m]} \;=\; \frac{e^{\rho\,\sigma_s\sigma_m} - 1}{e^{\sigma_m^2} - 1}\,. \tag{30}$$

Regardless of this, from a practical view there is the problem - as *Siegel* (1995) himself notes - that exchange options are not currently traded on the capital markets.

4.2 Estimating Beta Using Plain Vanilla Options

On incomplete markets, beta can be estimated implicitly with the valuation equations (13) and (20). As a result of the state of data typically given on the capital market, a two-stage process for estimating implicit beta is advisable. In a first step, expectations of the market participant with regard to the market index are estimated. Based on the valuation equation (20) for options on the market index, the sum of the squared relative differences between the empirical options prices $p(\tilde{X}_{cm})^*$ and theoretical options prices (20) is

[17] Inserting (16) and (17) in (5) results in (30).

minimized through selection of the parameter $\hat{\mu}_m$ and $\hat{\sigma}_m$,[18]

$$\min_{\mu_m,\sigma_m} \quad \sum_{j=1}^{J} \left(\frac{p\big(\tilde{X}_{cm}\big)^* - p\big(\tilde{X}_{cm}\big)}{p\big(\tilde{X}_{cm}\big)} \right)^2 . \tag{31}$$

Based on the parameters $\hat{\mu}_m$ and $\hat{\sigma}_m$, estimated in the first step, the parameters $\hat{\mu}_s$ and $\hat{\sigma}_s$ can be determined with the same method for any asset S ,

$$\min_{\mu_s,\sigma_s} \quad \sum_{j=1}^{J} \left(\frac{p\big(\tilde{X}_{c}\big)^* - p\big(\tilde{X}_{c}\big)}{p\big(\tilde{X}_{c}\big)} \right)^2 \tag{32}$$

Through the application of relative instead of absolute differences, it is avoided that in-the-money options influence estimations of the parameters much stronger than out-of-the-money options.

In the minimization, the correlation coefficient $\hat{\rho}$ cannot be estimated independently of the parameters $\hat{\mu}_s$ and $\hat{\sigma}_s$, as the CAPM equilibrium condition must be considered as an additional condition for the underlying asset,

$$p(\tilde{X}_s) = \frac{E\,[\tilde{X}_s] - \lambda \cdot Cov\,[\tilde{X}_s, \tilde{R}_m]}{1 + r_f^*} \qquad \text{where} \quad \lambda = \frac{E\,[\tilde{R}_m] - (1 + r_f^*)}{Var\,[\tilde{R}_m]} . \tag{33}$$

Following several conversions, inserting (1), (2), (3) and (4) in (33) results in

$$e^{\mu_s + \sigma^2/2} = \frac{e^{r_f}\left(e^{\sigma_m^2} - 1\right)}{\left(e^{\sigma_m^2} - 1\right) + \left(e^{-(\mu_m + \frac{1}{2}\sigma_m^2 - r_f)} - 1\right)\left(e^{\rho\,\sigma_m \sigma_s} - 1\right)} . \tag{34}$$

Implicit beta (5) of an asset S can be calculated with the estimated parameters.[19]

5 Summary and Conclusions

This article presents a technique with which beta can be estimated implicitly from the prices of plain vanilla options, without recourse to historical capital market data. The fundamental idea resembles that of *Latané and Rendleman* (1976) in the estimation of implicit volatilities from options prices: beta is estimated implicitly from options traded on the capital market, under the assumption of normally distributed rates of return based on the CAPM. Statistical errors that result from conventional regressive analysis of historical data, that beta values change through time, can hereby be avoided. Likewise, as in the estimation of implicit volatilities, the quality of the implicit beta depends on how well the options price model that is applied can explain the prices of the traded options.

[18] This technique is also applied by *Rubinstein* (1994) for the estimation of implicit risk-neutral density functions.

[19] In order to be able to calculate beta, (34) must be resolved accordingly,

$$\rho\,\sigma_m\,\sigma_s = \ln\left(1 + \frac{e^{2\mu_m + \sigma_m^2}\left(e^{\sigma_m^2} - 1\right)}{e^{\mu_m + \frac{1}{2}\sigma_m^2} - e^{r_f}} \cdot \frac{e^{\mu_s + \frac{1}{2}\sigma_s^2} - e^{r_f}}{e^{\mu_m + \frac{1}{2}\sigma_m^2 + \mu_s + \frac{1}{2}\sigma_s^2}}\right) , \tag{35}$$

and inserted in (5).

Appendix A: The Lognormal Distribution

The definition of density of normal distribution is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \tag{36}$$

$\Phi(\cdot)$ is the standard normal distribution ($\mu = 0$ and $\sigma = 1$). A variate is lognormally distributed if its natural logarithm is normally distributed. The definition of bivariate normal distribution is

$$f(x,y) = \frac{1}{2\pi\sqrt{\sigma_x^2\sigma_y^2(1-\rho^2)}} \cdot e^{-\frac{1}{2(1-\rho^2)}\left(\frac{(x-\mu_x)^2}{\sigma_x^2} - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right)}. \tag{37}$$

Two variates are bivariate lognormally distributed if their natural logarithms are bivariate normally distributed. In order to be able to calculate the moments of lognormally distributed variates (1), (2), (3) and (4), the simplifications of the following special integrals are required:

$$\int_a^\infty e^{cx}\, f(x)\, dx = e^{c\mu+\frac{1}{2}(c\sigma)^2} \cdot \Phi\left(\tfrac{-a+\mu+c\sigma^2}{\sigma}\right) \tag{38}$$

$$\int_{-\infty}^\infty \int_a^\infty e^{cx_2}\, f(x_1,x_2)\, dx_1\, dx_2 = e^{c\mu_2+\frac{1}{2}(c\sigma_2)^2} \cdot \Phi\left(\tfrac{-a+\mu_1+c\rho\,\sigma_1\sigma_2}{\sigma_1}\right) \tag{39}$$

$$\int_{-\infty}^\infty \int_a^\infty e^{x_1}e^{cx_2}\, f(x_1,x_2)\, dx_1\, dx_2 = e^{\mu_1+\frac{1}{2}\sigma_1^2+c\mu_2+\frac{1}{2}(c\sigma_2)^2+c\rho\,\sigma_1\sigma_2} \cdot \Phi\left(\tfrac{-a+\mu_1+\sigma_1^2+c\rho\,\sigma_1\sigma_2}{\sigma_1}\right) \tag{40}$$

In order to keep the proofs of (38), (39) and (40) concise in the following, it is convenient to use the conditional density. The definition of the conditional density is

$$f(x_2|x_1) = \frac{f(x_1,x_2)}{f(x_1)}.$$

If we apply this definition to the bivariate normal distribution, we get

$$f(x_2|x_1) = \frac{1}{\sqrt{2\pi\sigma_2^2(1-\rho^2)}} \cdot e^{-\frac{\left(x_2-\left(\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1)\right)\right)^2}{2\sigma_2^2(1-\rho^2)}}. \tag{41}$$

Note that the conditional density of the bivariate normal distribution equals the density of the normal distribution with the parameters

$$\mu_{x_2|x_1} = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1) \text{ und} \tag{42}$$

$$\sigma_{x_2|x_1}^2 = \sigma_2^2(1-\rho^2) \tag{43}$$

We next prove equation (38).

$$\begin{aligned}
\int_a^\infty e^{cx}\cdot\frac{1}{\sigma\sqrt{2\pi}}\cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx &= \int_a^\infty \frac{1}{\sigma\sqrt{2\pi}}\cdot e^{-\frac{x^2-2x\mu-2c\sigma^2x+\mu^2}{2\sigma^2}}\,dx \\
&= \int_a^\infty \frac{1}{\sigma\sqrt{2\pi}}\cdot e^{-\frac{x^2-2x(\mu+c\sigma^2)+\mu^2}{2\sigma^2}}\,dx \\
&= \int_a^\infty \frac{1}{\sigma\sqrt{2\pi}}\cdot e^{-\frac{x^2-2x(\mu+c\sigma^2)+(\mu+c\sigma^2)^2-(\mu+c\sigma^2)^2+\mu^2}{2\sigma^2}}\,dx \\
&= \int_a^\infty \frac{1}{\sigma\sqrt{2\pi}}\cdot e^{-\frac{(x-(\mu+c\sigma^2))^2}{2\sigma^2}}\cdot e^{-\frac{-(\mu+c\sigma^2)^2+\mu^2}{2\sigma^2}}\,dx \\
&= e^{c\mu+\frac{1}{2}(c\sigma)^2}\int_a^\infty \frac{1}{\sigma\sqrt{2\pi}}\cdot e^{-\frac{(x-(\mu+c\sigma^2))^2}{2\sigma^2}}\,dx
\end{aligned}$$

Equation (38) follows with $1-\Phi\big(\frac{a-(\mu+c\sigma^2)}{\sigma}\big) = \Phi\big(\frac{-a+\mu+c\sigma^2}{\sigma}\big)$. The proof for equation (39) is given under consideration of the conditional density indicated above,

$$\int_{-\infty}^\infty\int_a^\infty e^{cx_2}\,f(x_1,x_2)\,dx_1\,dx_2 = \int_a^\infty\left[\int_{-\infty}^\infty e^{cx_2}\,f(x_2|x_1)\,dx_2\right]f(x_1)\,dx_1\ .$$

The integral in brackets can be interpreted in that the expected value and the variance according to (42) and (43) are transformed and the equation (38) is subsequently used,

$$\begin{aligned}
\int_a^\infty\left[\int_{-\infty}^\infty e^{cx_2}\,f(x_2|x_1)\,dx_2\right]f(x_1)\,dx_1 &= \int_a^\infty\left[e^{c\left(\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1)\right)+\frac{1}{2}c^2\left(\sigma_2^2(1-\rho^2)\right)}\right]f(x_1)\,dx_1 \\
&= e^{c\mu_2-c\rho\frac{\sigma_2}{\sigma_1}\mu_1+\frac{1}{2}c^2\sigma_2^2-\frac{1}{2}c^2\sigma_2^2\rho^2}\int_a^\infty e^{c\rho\frac{\sigma_2}{\sigma_1}x_1}f(x_1)\,dx_1\ .
\end{aligned}$$

If we define the helping variable $c^* := c\rho\frac{\sigma_2}{\sigma_1}$, we arrive at the equation (39) after application of (38) and shortening of the terms in exponents. The proof for equation (40) can be shown analogously,

$$\begin{aligned}
\int_{-\infty}^\infty\int_a^\infty e^{cx_2}e^{x_1}\,f(x_1,x_2)\,dx_1\,dx_2 &= \int_a^\infty\left[\int_{-\infty}^\infty e^{cx_2}\,f(x_2|x_1)\,dx_2\right]e^{x_1}f(x_1)\,dx_1 \\
&= e^{c\mu_2-c\rho\frac{\sigma_2}{\sigma_1}\mu_1+\frac{1}{2}c^2\sigma_2^2-\frac{1}{2}c^2\sigma_2^2\rho^2}\int_a^\infty e^{c\rho\frac{\sigma_2}{\sigma_1}x_1+x_1}f(x_1)\,dx_1\ .
\end{aligned}$$

If we define the helping variable $c^{**} := c\rho\frac{\sigma_2}{\sigma_1}+1$, we arrive at the desired result (40) after repeated application of equation (38) and shortening of the terms in exponents.

Appendix B: Option Pricing Using the CAPM

In order to calculate the expected value of a call (7), we use equation (38),

$$\begin{aligned}
E\,[\tilde{X}_c] &= \int\limits_{-\infty}^{\infty} \max\Big(p(\tilde{X}_s)\,e^{r_s} - K, 0\Big)\; f(r_s)\,dr_s \\
&= p(\tilde{X}_s) \int\limits_{\ln\left(K/P(\tilde{X}_s)\right)}^{\infty} e^{r_s}\, f(r_s)\,dr_s \;-\; K \int\limits_{\ln\left(K/P(\tilde{X}_s)\right)}^{\infty} f(r_s)\,dr_s \\
&= p(\tilde{X}_s) e^{\mu_s + \frac{1}{2}\sigma_s^2} \cdot \Phi\left(\tfrac{\ln(p(\tilde{X}_s)/K) + \mu_s + \sigma_s^2}{\sigma_s}\right) - K\,\Phi\left(\tfrac{\ln(p(\tilde{X}_s)/K) + \mu_s}{\sigma_s}\right). \qquad (44)
\end{aligned}$$

We can simplify the calculation of the covariance through application of the decomposition theorem. From equations (39) and (40) result

$$\begin{aligned}
E\,[\tilde{X}_c \tilde{R}_m] &= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \max(p(\tilde{X}_s)\,e^{r_s} - K, 0)\cdot\, e^{r_m}\, f(r_s, r_m)\,dr_m\,dr_s \\
&= \int\limits_{-\infty}^{\infty} \int\limits_{\ln\left(K/P(\tilde{X}_s)\right)}^{\infty} (p(\tilde{X}_s)\,e^{r_s} - K)\cdot\, e^{r_m}\, f(r_s, r_m)\,dr_m\,dr_s \\
&= p(\tilde{X}_s)\cdot \int\limits_{-\infty}^{\infty} \int\limits_{\ln\left(K/P(\tilde{X}_s)\right)}^{\infty} e^{r_s} e^{r_m}\, f(r_s, r_m)\,dr_m\,dr_s - K\cdot \int\limits_{-\infty}^{\infty} \int\limits_{\ln\left(K/P(\tilde{X}_s)\right)}^{\infty} e^{r_m}\, f(r_s, r_m)\,dr_m\,dr_s \\
&= p(\tilde{X}_s)\cdot\; e^{\mu_s + \frac{1}{2}\sigma_s^2 + \mu_m + \frac{1}{2}\sigma_m^2 + \rho\,\sigma_s\sigma_m} \cdot \Phi\left(\frac{-\ln\left(K/p(\tilde{X}_s)\right) + \mu_s + \sigma_s^2 + \rho\,\sigma_s\sigma_m}{\sigma_s}\right) \\
&\quad - K\cdot\; e^{\mu_m + \frac{1}{2}\sigma_m^2} \cdot \Phi\left(\frac{-\ln\left(K/p(\tilde{X}_s)\right) + \mu_s + \rho\,\sigma_s\sigma_m}{\sigma_s}\right)\;. \qquad (45)
\end{aligned}$$

Following the decomposition theorem, we arrive at the covariance (8) with (44) und (45), after elementary conversions.

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